A New Solution for TE Plane-Wave Scattering from a Symmetric Double-Strip Grating Composed of Equal Strips

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Abstract—**The paper presents a new rigorous solution for the problem of TE plane-wave scattering from a periodic planar symmetric double-strip grating, i.e., the grating which has two equal strips per unit cell. The grating is placed at a dielectric interface and is assumed to be perfectly conductive and infinite in length and width. The formulation is based on a multimode equivalent network representation and the relevant integral equation defined on two separate intervals is rigorously solved by reducing to two simpler equations with known solutions. From this a new simple analytic expression is obtained for the coupling matrix elements which involves no integration. Some computations based on this new expression are carried out and the results are compared to those obtained by the Riemann–Hilbert method and also to some of the previously obtained single-strip results in the limiting case.**

I. Introduction

A RIGOROUS multimode network formulation for the problem of transverse electric (TE) or transverse magnetic (TM) plane-wave scattering from a planar periodic single-strip metal grating at a dielectric interface was proposed in [1]. In that formulation the grating is represented by a mutual coupling matrix whose elements correspond to coupling between the different space harmonics, excited by the grating. The incident wave as well as the reflected and the transmitted harmonics are modeled by transmission lines. The coupling matrix is in the form of an impedance matrix for the aperture formulation, in which the electric field between the strips is used as the unknown, and is an admittance matrix in the obstacle formulation, which uses the current on the strips as the unknown. Besides these two formulations there are two possible wave polarizations, TE and TM. The coupling matrix elements depend both on the polarization and the formulation chosen. In each of the four possible combinations the coupling matrix elements are related to an integral over the grating unit cell of an unknown function which has to be found from an integral equation. This integral equation has one particular form for the TM-obstacle and TE-aperture cases, and a different form for the TM-aperture and TE-obstacle cases. A novel rigorous solution of the integral equation for a single-strip grating was presented in [2] for the cases of TM-obstacle and TE-aperture formulations. From this, an analytic expression for the coupling matrix elements was obtained in the form of certain double integrals which have to be evaluated numerically.

In [3] a new simple analytic expression for the coupling matrix elements in the single-strip case was derived, which compared to the corresponding expression in [2] involves no integration. In this approach the starting integral equation was the one for the TM-aperture and TE-obstacle formulations.

The method of [2] was extended in [4] to treat the case of a double-strip grating, i.e., the grating which has two strips per unit cell. Although the method is basically the same as for the single-strip grating, the addition of the extra strip introduces significant mathematical difficulties in the solution procedure.

In this paper the method from [3] is extended to the problem of TE plane-wave scattering from a symmetric double-strip grating, which has equal strips. The grating is placed at a dielectric interface, and is assumed to be perfectly conductive and infinite in two directions. A suitable formulation for this problem is the TE-obstacle formulation, and the relevant integral equation is the modified integral equation from [3]. The modification consists in removing a symmetrically spaced interval from the basic domain of integration. The modified integral equation is solved by reducing to two simpler equations with known solutions. One of them is Carleman's integral equation, and the other is a Cauchy-type singular integral equation. Using the solution of the integral equation, a new simple analytic expression for the coupling matrix elements is obtained. This expression turns out to be similar in form to the corresponding single-strip expression in [3].

Some computations based on this new expression have been carried out and the results are compared to those obtained by the Riemann–Hilbert method. Also a comparison is made with previously obtained results for the single-strip grating as a limiting case.

II. Formulation of the Problem and the Solution of the Relevant Integral Equation

Geometry of the problem is shown in Fig. 1. A plane, TE polarized wave is incident at an angle θ upon a planar periodic symmetric double-strip grating at a dielectric interface. The grating has two equal strips per period p, the distance between the strips within the grating unit cell is d_1, and the strip width is $(d - d_1)/2$. An appropriate formulation is the TE-obstacle

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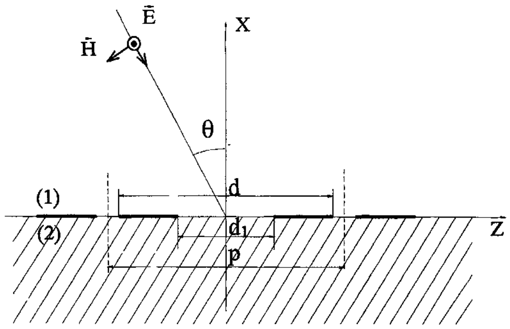

Fig. 1. A symmetric periodic double-strip metal grating at a dielectric interface. The grating has two equal strips per unit cell. A plane TE polarized wave is incident at an angle.

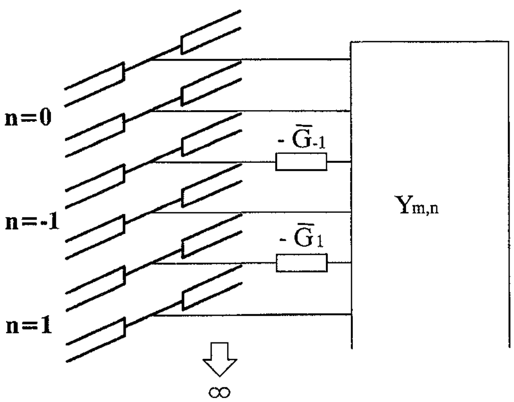

Fig. 2. The rigorous equivalent network related to the structure shown in Fig. 1.

formulation for which the equivalent network representation is shown in Fig. 2. The quantities in this figure are given in [1].

The coupling matrix elements (which are admittances in this case) are to be found from

$$Y_{mn} = \frac{1}{2B} \,⨎_{-A}^{A} F_n(\xi) e^{jm\xi}\, d\xi \tag{1}$$

where B is a constant [1] and

$$A = \frac{\pi d}{p}.$$

The unknown function $F_n(\xi)$ which appears in (1) satisfies the integral equation

$$⨎_{-A}^{A} F_n(\xi') \ln \frac{1}{2\left|\sin \dfrac{\xi - \xi'}{2}\right|}\, d\xi' = e^{-jn\xi}. \tag{2}$$

In (1) and (2) the equal sign on the integral sign means that the interval $(-a,\, a)$ where

$$a = \frac{\pi d_1}{p}$$

is removed from the basic domain of integration, i.e., the integration is performed on the two separate intervals $(-A,\, -a)$ and $(a,\, A)$.

Integral (2) is equivalent to the following two equations

$$⨎_{-A}^{A} \varphi_n(\xi') \ln \frac{1}{2\left|\sin \dfrac{\xi - \xi'}{2}\right|}\, d\xi' = \cos n\xi \tag{3}$$

and

$$⨎_{-A}^{A} \psi_n(\xi') \ln \frac{1}{2\left|\sin \dfrac{\xi - \xi'}{2}\right|}\, d\xi' = -\sin n\xi \tag{4}$$

where

$$\varphi_n(\xi) = \operatorname{Re} F_n(\xi) \tag{5}$$

$$\psi_n(\xi) = \operatorname{Im} F_n(\xi). \tag{6}$$

Consider first (3). It can be rewritten as

$$\int_{-A}^{-a} \varphi_n(\xi') \ln \frac{1}{2\left|\sin \dfrac{\xi - \xi'}{2}\right|}\, d\xi' + \int_{a}^{A} \varphi_n(\xi') \ln \frac{1}{2\left|\sin \dfrac{\xi - \xi'}{2}\right|}\, d\xi' = \cos n\xi. \tag{7}$$

It is easy to see from (7) that the unknown function $\varphi_n(\xi)$ must be an even function, since the cosine on the right-hand side is an even function. Then, by making a change of variables $\xi' \to -\xi'$ in the first integral, after some elementary transformations, (7) simplifies to

$$\int_{a}^{A} \varphi_n(\xi') \ln \frac{1}{2|\cos \xi' - \cos \xi|}\, d\xi' = \cos n\xi$$

which in turn, by a change of variables

$$\cos \xi' = \frac{\theta'}{2}, \qquad \cos \xi' = \frac{\theta}{2}$$

reduces to

$$\int_{2u}^{2v} \Phi_n(\theta') \ln |\theta' - \theta|\, d\theta' = -T_n \frac{\theta}{2} \tag{8}$$

where

$$\Phi_n(\theta) = \frac{\varphi_n\left(\arccos \dfrac{\theta}{2}\right)}{\sqrt{4 - \theta^2}} \tag{9}$$

$$u = \cos A$$

$$v = \cos a$$

and

$$T_n(x) = \cos(n \arccos x)$$

is the Chebyshev polynomial of the first kind.

Equation (8) is the known Carleman's equation. Its closed-form solution is [5]

$$\Phi_n(\theta) = \frac{1}{\pi^2\sqrt{(\theta-2u)(2v-\theta)}} \cdot \left[-\text{v.p.}\int_{2u}^{2v} \frac{\sqrt{(\theta'-2u)(2v-\theta')}\,T_n'\left(\frac{\theta'}{2}\right)\frac{1}{2}\,d\theta'}{\theta'-\theta}\right] - \frac{1}{\ln\frac{v-u}{2}}\int_{2u}^{2v}\frac{T_n\left(\frac{\theta'}{2}\right)d\theta'}{\sqrt{(\theta'-2u)(2v-\theta')}} \tag{10}$$

where v.p. means principal value.

Substituting in (10)$\Phi_n(\theta)$ from (9) and returning to the old variables ξ, ξ', yields

$$\varphi_n(\xi) = \frac{-\sin\xi}{\pi^2\sqrt{(\cos\xi-u)(v-\cos\xi)}} \cdot \left[\text{v.p.}\int_a^A \frac{\sqrt{(\cos\xi'-u)(v-\cos\xi')}\,n\sin n\xi'\,d\xi'}{\cos\xi'-\cos\xi}\right] + \frac{1}{\ln\frac{v-u}{2}}\int_a^A \frac{\cos n\xi'\sin\xi'\,d\xi'}{\sqrt{(\cos\xi'-u)(v-\cos\xi')}} \tag{11}$$

The integrals appearing in (11) are evaluated in Appendix I. They are

$$\text{v.p.}\int_a^A \frac{\sqrt{(\cos\xi'-u)(v-\cos\xi')}n\sin n\xi'\,d\xi'}{\cos\xi'-\cos\xi} = -\frac{\pi|n|}{2\sin\xi}\sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}(u,v)\sin(p+1)\xi \tag{12}$$

$$\int_a^A \frac{\cos n\xi'\sin\xi'\,d\xi'}{\sqrt{(\cos\xi'-u)(v-\cos\xi')}} = \frac{\pi}{2}\left[Q_n(u,v)-Q_{n-2}(u,v)\right] \tag{13}$$

where $\rho_k(u, v)$ and $Q_k(u, v)$ are some polynomials in two variables defined in Appendices I and II, respectively.

From (11)–(13) the final solution of (3) is obtained as 12

$$\varphi_n(\xi) = \frac{-\sin\xi}{2\pi\ln\frac{v-u}{2}\sqrt{(\cos\xi-u)(v-\cos\xi)}} \cdot \left[Q_n(u,v)-Q_{n-2}(u,v)\right] + \frac{|n|}{2\pi\sqrt{(\cos\xi-u)(v-\cos\xi)}} \cdot \sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}(u,v)\sin(p+1)\xi. \tag{14}$$

Consider now (4). Again, the interval of integration is broken up into two intervals $(-A, -a)$ and (a, A), and a change of variables $\xi' \to -\xi'$ is made in the first integral. Then, taking into account that the unknown function $\psi_n(\xi)$ is an odd function, (4) can be rewritten as

$$\int_a^A \psi_n(\xi')\ln\left|\frac{\sin\frac{\xi+\xi'}{2}}{\sin\frac{\xi-\xi'}{2}}\right|d\xi' = -\sin n\xi \tag{15}$$

which upon differentiation with respect to ξ becomes

$$\int_a^A \psi_n(\xi')\frac{-\sin\xi'}{\cos\xi'-\cos\xi}\,d\xi' = -n\cos n\xi$$

where the integral has to be understood in the sense of principal value. Introducing here new variables

$$\cos\xi' = \theta', \quad \cos\xi = \theta$$

yields

$$\int_u^v \Psi_n(\theta')\frac{d\theta'}{\theta'-\theta} = nT_n(\theta) \tag{16}$$

where

$$\Psi_n(\theta) = \psi_n(\arccos\theta). \tag{17}$$

Equation (16) is a Cauchy-type singular equation whose closed-form solution, unbounded for $\theta = u$ and $\theta = v$ is given by [5]

$$\Psi_n(\theta) = -\frac{1}{\sqrt{(\theta-u)(v-\theta)}} \cdot \left[\frac{1}{\pi^2}\text{v.p.}\int_u^v \frac{\sqrt{(\theta'-u)(v-\theta')}\,nT_n(\theta')d\theta'}{\theta'-\theta} + c_n\right] \tag{18}$$

where $c_n = c_n(u, v)(n = \pm1, \pm2, \cdots)$ are constants to be determined. It is obvious from (16) and (18) that $c_{-n} = -c_n$ since Ψ_n should change sign when n changes sign.

Substituting in (18) $\Psi_n(\theta)$ from (17) and returning to the old variables ξ, ξ', yields

$$\psi_n(\xi) = -\frac{1}{\sqrt{(\cos\xi-u)(v-\cos\xi)}} \cdot \left[\frac{1}{\pi^2}\text{v.p.}\int_a^A \frac{\sqrt{(\cos\xi'-u)(v-\cos\xi')}\,n\cos n\xi'\sin\xi'\,d\xi'}{\cos\xi'-\cos\xi} + c_n\right]. \tag{19}$$

The integral in (19) is evaluated in Appendix I. It is as shown in (20) at the bottom of the following page. Substituting this into (19) gives

$$\psi_n(\xi) = \frac{n}{2\pi\sqrt{(\cos\xi - u)(v - \cos\xi)}} \cdot \left[\frac{1}{2}\overline{\rho}_{|n|+1} + \sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}\cos(p+1)\xi\right] - \frac{c_n}{\sqrt{(\cos\xi - u)(v-\cos\xi)}} \tag{21}$$

where for $n = \pm 1$, the first term inside the brackets should be replaced by $\frac{1}{2} + \frac{1}{2}\overline{\rho}_2$.

If new constants $C_n = C_n(u, v)$ are defined by

$$C_n = \begin{cases} \frac{1}{2} + \frac{1}{2}\overline{\rho}_2 - 2\pi c_1, & n = \pm 1 \\ \frac{1}{2}\overline{\rho}_{|n|+1} - \frac{2\pi c_n}{n}, & n \neq \pm 1 \end{cases} \tag{22}$$

the solution of (4) given by (21) can be rewritten as

$$\psi_n(\xi) = \frac{n}{2\pi\sqrt{(\cos\xi - u)(v-\cos\xi)}} \cdot \left[C_{|n|}(u, v) + \sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}(u, v)\cos(p+1)\xi\right], \quad a < \xi < A. \tag{23}$$

The constants $C_n (n = \pm 1, \pm 2, \cdots)$ defined by (22) do not depend on the sign of n, i.e., $C_{-n} = C_n$ which is indicated in (23) by writing $|n|$ in the subscript of C.

The constants $C_{|n|}$ can be evaluated numerically as follows. The solution $\psi_n(\xi)$ given by (23) is substituted into (15) thereby giving an identity for $a < \xi < A$. Choosing an arbitrary ξ from this interval and numerically evaluating the integral in (15) one obtains an equation for determining $C_{|n|}$.

III. Determination of the Coupling Matrix Elements

From (1), (5), and (6) the coupling matrix elements are

$$Y_{mn} = \frac{1}{2B}\left\{\left(\int_{-A}^{-a} + \int_a^A\right)[\varphi_n(\xi) + j\psi_n(\xi)] \cdot (\cos m\xi + j\sin m\xi)\, d\xi\right\}. \tag{24}$$

Since $\varphi_n(\xi)$ is an even function and $\psi_n(\xi)$ is an odd function (24) is simplified to

$$Y_{mn} = \frac{1}{2B}\cdot 2\int_a^A [\varphi_n(\xi)\cos m\xi - \psi_n(\xi)\sin n\xi]\, d\xi.$$

Substituting here $\varphi_n(\xi)$ and $\psi_n(\xi)$ from (14) and (23) and using (A10) gives

$$Y_{mn} = \frac{1}{2B}\left[-\frac{(Q_n - Q_{n-2})(Q_m - Q_{m-2})}{2\ln\frac{v-u}{2}} + \frac{|n|}{2}\sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}(Q_{m+p} - Q_{m-p-2}) - nC_{|n|}Q_{m-1} - \frac{n}{2}\sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}(Q_{m+p} - Q_{-m+p})\right] \tag{25}$$

where the arguments u and v are omitted.

Finally, considering in (25) the cases $n \geq 1$ and $n \leq -1$ separately, and using (A11) yields

$$Y_{mn} = \frac{1}{2B} \cdot \left[-\frac{(Q_n - Q_{n-2})(Q_m - Q_{m-2})}{2\ln\frac{v-u}{2}} - nC_{|n|}Q_{m-1} + n\begin{Bmatrix} \sum_{p=0}^{n}\overline{\rho}_{n-p}Q_{-m+p}, & n \geq 1 \\ -\sum_{p=0}^{-n}\overline{\rho}_{-n-p}Q_{m+p}, & n \leq -1 \end{Bmatrix}\right]. \tag{26}$$

Relation (26) is a new simple analytic expression for the coupling matrix elements related to TE plane-wave scattering from a symmetric double-strip grating which has equal strips. Note the similarity of (26) with the corresponding expression in [3] for the single strip grating.

IV. Numerical Results

Using the new expression for the coupling matrix elements and the equivalent network shown in Fig. 2, some computations have been carried out for TE plane-wave scattering from a symmetric double-strip grating shown in Fig. 1.

Fig. 4 shows the transmission coefficient in the lowest ($n = 0$) mode versus the relative period p/λ_0 for different values of parameter d/p. The grating is in free space, and the other parameters are $\theta = 0$ and $d_1/p = 0.5$. In these computations modes of order up to ± 4 are included. The values obtained by the Riemann–Hilbert method [6] are indicated by dots. As can

$$\int_a^A \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\; n\cos n\xi'\sin\xi'\, d\xi'}{\cos\xi - \cos\xi} = \begin{cases} -\frac{n\pi}{2}\left(\frac{1}{2} + \frac{1}{2}\overline{\rho}_{|n|+1} + \sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}\cos(p+1)\xi, \quad n = \pm 1\right) \\ -\frac{n\pi}{2}\left(\frac{1}{2}\overline{\rho}_{|n|+1} + \sum_{p=0}^{|n|}\overline{\rho}_{|n|-p}\cos(p+1)\xi, \quad n \neq \pm 1\right) \end{cases} \tag{20}$$

be seen, the agreement is very good. If $d/p = 1$, the double-strip grating from Fig. 1 becomes a single-strip grating, so it should be expected that the curve with $d/p = 0.99$ be very close to the corresponding single-strip curve, obtained by using the new solution from [3]. Indeed, the two curves are almost indistinguishable.

Fig. 5 shows the normalized transmitted power versus the relative period for $\theta = 15°$, $\varepsilon_r^{(2)} = 2$, and $d_1/p = 0.5$, and for different values of parameter d/p. Modes of order up to ± 6 are included. Again, the curve with $d/p = 0.99$ practically coincides with the corresponding single-strip curve from [3].

V. Conclusion

Starting from the multimode network representation for the problem of TE plane-wave scattering from a symmetric periodic planar double-strip grating composed of equal strips, and placed at a dielectric interface, a new simple analytic expression for the coupling matrix elements has been derived. This expression involves no integration and is similar in form to the corresponding single-strip expression obtained earlier. The relevant integral equation is defined on two separate symmetrically spaced intervals and is solved by reducing to two simpler equations with known solution. Comparison with the Riemann–Hilbert solution shows a very good agreement. Also, the results obtained with the present method coincide with the previously obtained single-strip results in the limiting case.

Appendix I

The integrals encountered in (11) and (19) which have to be evaluated are

$$I_1 = \int_a^A \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, n \sin n\xi'\, d\xi'}{\cos\xi' - \cos\xi}$$

$$I_2 = \int_a^A \frac{\cos n\xi' \sin\xi'\, d\xi'}{\sqrt{(\cos\xi' - u)(v - \cos\xi')}}$$

$$I_3 = \int_a^A \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, n \cos n\xi' \sin\xi'\, d\xi'}{\cos\xi' - \cos\xi}.$$

The integrals I_1 and I_3 are understood in the sense of principal value. The symbol v.p. indicating this is omitted throughout the Appendix.

Consider first the integral I_1. It is invariant with respect to the sign of n so that positive values of n are assumed. The following relation

$$\int_a^A \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, n \sin n\xi'\, d\xi'}{\cos\xi' - \cos\xi} = -\frac{1}{4\cos\frac{\xi}{2}} \oint_{-A}^{A} \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, n \sin n\xi'\, d\xi'}{\sin\frac{\xi' - \xi}{2}\left|\sin\frac{\xi'}{2}\right|}$$

can be easily verified by writing $\oint_{-A}^{A} = \int_{-A}^{-a} + \int_a^A$ and by making a change of variables $\xi' \to -\xi'$ in the first integral. Therefore

$$I_1 = -\frac{n}{4\cos\frac{\xi}{2}} \operatorname{Im} \oint_{-A}^{A} \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, e^{jn\xi'}\, d\xi'}{\sin\frac{\xi' - \xi}{2}\sqrt{\frac{1 - \cos\xi'}{2}}} \tag{A1}$$

The integral on the right-hand side of (A1) can be evaluated by the residue techniques. It is first transformed by putting $e^{j\xi'} = z$, $e^{j\xi} = z_0$ yielding

$$\oint_{-A}^{A} \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, e^{jn\xi'}\, d\xi'}{\sin\frac{\xi' - \xi}{2}\sqrt{\frac{1 - \cos\xi'}{2}}} = -2\sqrt{z_0} \int_\Gamma \frac{\sqrt{(z^2 - 2uz + 1)(z^2 - 2vz + 1)}\, z^{n-1}\, dz}{(z - z_0)(z - 1)}. \tag{A2}$$

The contour of integration consists of two symmetrically spaced arcs of the unit circle $|z| = 1$. The end points of the arcs are β^*, α^*, α, and β where $\alpha = e^{ja}$, $\beta = e^{jA}$ and the star means complex conjugate value. These points are the branch points of the two-valued function

$$w(z) = \sqrt{(z^2 - 2uz + 1)(z^2 - 2vz + 1)}. \tag{A3}$$

This function becomes single-valued by making two cuts along the unit circle—one between the points α^* and β^*, the other between the points α and β. The condition $w(0) = 1$ specifies one of the two possible branches, and the contour of integration goes along the inner lips of the two cuts. Let Γ' be the closed contour going along the inner and outer lips of the cuts in the clockwise directions (Fig. 3). Then, the integral along Γ is equal to one half of the integral along Γ', since the square root changes sign on the outer lips. The integral along the closed contour Γ' can be evaluated by means of the residues at the two singular points $z = 0$ and $z = \infty$. Therefore

$$\oint_{-A}^{A} \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, e^{jn\xi'}\, d\xi'}{\sin\frac{\xi' - \xi}{2}\sqrt{\frac{1 - \cos\xi'}{2}}} = -2\sqrt{z_0}\,\tfrac{1}{2} \cdot 2\pi j[\operatorname{Re} s f(1) + \operatorname{Re} s f(\infty)] \tag{A4}$$

where $f(z)$ is the function under the integral sign on the right-hand side of (A2). The residue at $z = 1$ is easily found to be

$$\operatorname{Re} s f(1) = \frac{2\sqrt{(1-u)(1-v)}}{1 - z_0}. \tag{A5}$$

To find the residue at $z = \infty$ the function $f(z)$ has to be developed into the Laurent series for $|z| > 1$. Let $\overline{\rho}_k(u, v)$ $(k = 0, 1, 2, \cdots)$ be the coefficients in the Taylor expansion of the function $w(z)$ defined by (A3)

$$w(z) = \sum_{k=0}^{\infty} \overline{\rho}_k(u, v) z^k, \quad |z| < 1. \tag{A6}$$

These coefficients are found in Appendix II. For $|z| > 1$, one can replace z by $1/z$ and then apply (A6). The result is

$$w(z) = z^2 \sum_{k=0}^{\infty} \overline{\rho}_k(u, v) z^{-k}, \quad |z| > 1.$$

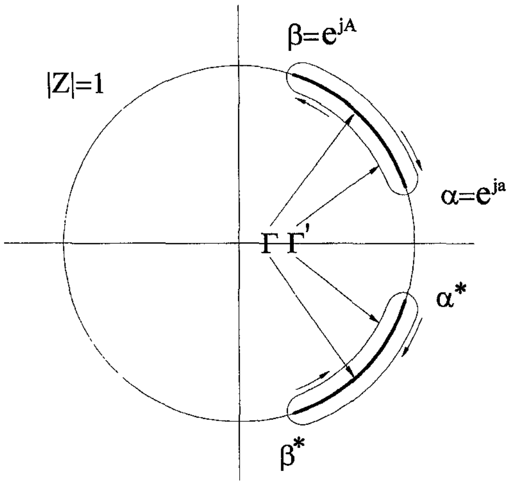

Fig. 3. The contours of integration Γ and Γ′ in the complex plane.

Now

$$\frac{w(z)z^{n-1}}{(z-z_0)(z-1)} = z^2 \sum_{k=0}^{\infty} \overline{\rho}_k z^{-k} z^{n-1} \cdot \frac{1}{z} \sum_{k=0}^{\infty} z^{-k} \frac{1}{z} \sum_{k=0}^{\infty} z_0^k z^{-k} = z^{n-1} \sum_{k=0}^{\infty} \beta_k z^{-k}, \quad |z| > 1 \quad \text{(A7)}$$

where

$$\beta_k = \sum_{p=0}^{k} \overline{\rho}_{k-p} \sum_{i=0}^{p} z_0^i.$$

The residue at $z = \infty$ is the coefficient of $1/z$ in the Laurent series (A7) with the opposite sign. It is

$$\operatorname{Re} s f(\infty) = -\beta_n = -\sum_{p=0}^{n} \overline{\rho}_{n-p} \sum_{i=0}^{p} z_0^i. \quad \text{(A8)}$$

From (A4), (A5), and (A8)

$$\fint_{-A}^{A} \frac{\sqrt{(\cos\xi' - u)(v - \cos\xi')}\, e^{jn\xi'}\, d\xi'}{\sin\dfrac{\xi' - \xi}{2}\sqrt{\dfrac{1-\cos\xi'}{2}}} = -2\pi j\sqrt{z_0}\left[\frac{2\sqrt{(1-u)(1-v)}}{1-z_0} - \sum_{p=0}^{n} \overline{\rho}_{n-p} \sum_{i=0}^{p} z_0^i\right].$$

Replacing here z_0 by $e^{j\xi}$, taking the imaginary parts and substituting into (A1) yields

$$I_1 = -\frac{\pi n}{2\cos\dfrac{\xi}{2}} \sum_{p=0}^{n} \overline{\rho}_{n-p}(u, v) \sum_{i=0}^{p} \cos\left(i + \frac{1}{2}\right)\xi.$$

If the nominator and the denominator on the right-hand side are multiplied by $\sin(\xi/2)$, after some elementary transformations I_1 is simplified to the form shown in (12) (n is replaced by $|n|$ since I_1 is invariant with respect to the sign of n).

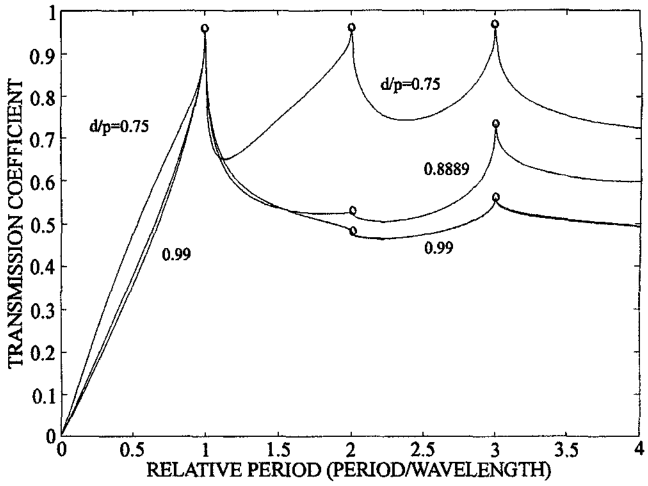

Fig. 4. The transmission coefficient in the lowest ($n = 0$) mode versus the relative period. The parameters have the following values: $d_1/p = 0.5, \theta = 0$, and $\varepsilon_r^{(1)} = \varepsilon_r^{(2)} = 1$. The parameter d/p is varied. The values obtained by the Riemann–Hilbert method are shown by dots. The curve with $d/p = 0.99$ practically coincides with the corresponding single-strip curve.

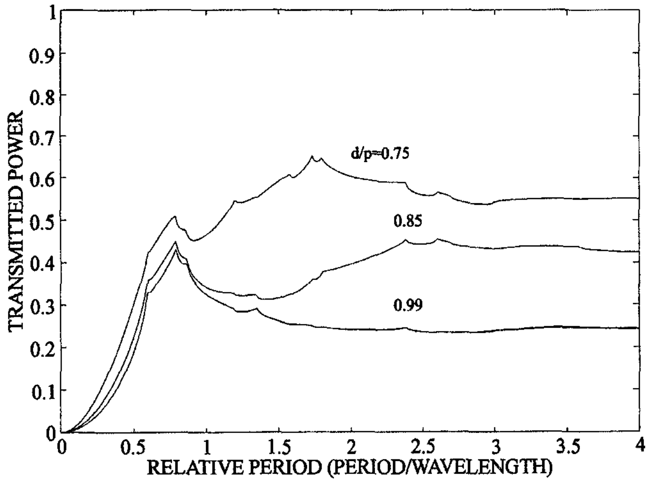

Fig. 5. The transmitted power in the lowest mode (normalized to the incident power) versus the relative period. The parameters are: $d_1/p = 0.5$, $\theta = 15°$, $\varepsilon_r^{(1)} = 1$, and $\varepsilon_r^{(2)} = 2$. The parameter d/p is varied. The curve with $d/p = 0.99$ is practically indistinguishable from the corresponding single-strip curve.

Before proceeding to the evaluation of I_2 it is useful to define polynomials $Q_k(u, v)$ as the coefficients in the Taylor expansion of the function $1/w(z)$ for $|z| < 1$

$$\frac{1}{w(z)} = \sum_{k=0}^{\infty} Q_k(u, v) z^k, \quad |z| < 1. \quad \text{(A9)}$$

Since

$$\frac{1}{\sqrt{z^2 - 2uz + 1}} = \sum_{k=0}^{\infty} P_k(u) z^k, \quad |z| < 1$$

and

$$\frac{1}{\sqrt{z^2 - 2vz + 1}} = \sum_{k=0}^{\infty} P_k(v) z^k, \quad |z| < 1$$

the coefficients Q_k in (A9) can be found as

$$Q_k(u, v) = \sum_{p=0}^{k} P_{k-p}(u)P_p(v).$$

These polynomials have an integral representation which enables to define them for negative values of n. This representation is derived in [6] and has the following form

$$Q_n(u, v) = \frac{1}{\pi}\int_a^A \frac{\sin(n+1)\varphi \, d\varphi}{\sqrt{(\cos\varphi - u)(v - \cos\varphi)}}. \quad \text{(A10)}$$

Obviously, from (A10)

$$\begin{aligned} Q_{-1}(u, v) &\equiv 0 \\ Q_{-n}(u, v) &= -Q_{n-2}(u, v), \quad n \geq 2. \end{aligned} \quad \text{(A11)}$$

The first few polynomials Q_n are

$$\begin{aligned} Q_0 &= 1, \\ Q_1 &= u + v, \\ Q_2 &= \tfrac{3}{2}(u^2 + v^2) + uv - 1, \\ Q_3 &= \tfrac{5}{2}(u^3 + v^3) + \tfrac{3}{2}uv(u+v) - 2(u+v). \end{aligned}$$

Now, the integral I_2 can be evaluated. It is first transformed as

$$I_2 = \frac{1}{2}\int_a^A \frac{\sin(n+1)\xi' - \sin(n-1)\xi'}{\sqrt{(\cos\xi' - u)(v - \cos\xi')}} \, d\xi.$$

Then, using (A10) yields the result shown in (13).

Finally consider the integral I_3. Again, positive values of n are assumed since for negative values I_3 only changes sign. By a simple trigonometric identity I_3 can be rewritten as the difference of two integrals which both have the same form as the integral I_1. Therefore using (12) followed by some elementary transformations yields the result shown in (20).

APPENDIX B

In this appendix, the polynomials $\overline{\rho}_k(u, v)$ $(k = 0, 1, 2, \cdots)$ are evaluated. One has [3]

$$\sqrt{z^2 - 2uz + 1} = \sum_{k=0}^{\infty} \rho_k(u) z^k, \quad |z| < 1 \quad \text{(B1)}$$

where $\rho_0 = 1$, $\rho_1(u) = -u$ and

$$\rho_n(u) = P_n(u) - 2uP_{n-1}(u) + P_{n-2}(u), \quad n \geq 2$$

where P_k's are the Legendre polynomials.

Similarly

$$\sqrt{z^2 - 2vz + 1} = \sum_{k=0}^{\infty} \rho_k(v) z^k, \quad |z| < 1. \quad \text{(B2)}$$

Multiplication of (B1) and (B2) and comparison with (A6) yields

$$\overline{\rho}_k(u, v) = \sum_{p=0}^{k} \rho_{k-p}(u)\rho_p(v)$$

The first few ρ_k's are

$$\begin{aligned} \overline{\rho}_0 &= 1, \\ \overline{\rho}_1 &= -(u+v), \\ \overline{\rho}_2 &= -\tfrac{1}{2}(u^2 + v^2) + uv + 1, \\ \overline{\rho}_3 &= -\tfrac{1}{2}(u^3 + v^3) + \tfrac{1}{2}uv(u+v). \end{aligned}$$

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